on $S$, and

$$(G_{22} + G_{33})\ \varphi,_r|_{r=R} + (G_{22} - G_{33})\{\varphi,_r|_{r=R} \cos 2\theta$$

$$- [R + (1/R)\varphi,_\theta]|_{r=R} \sin 2\theta\} = 0$$

on $\partial S$.

The solution sought is of the form

$$\varphi = \phi(r)\psi(\theta)$$

with $\psi(\theta),_{\theta\theta} = -4\psi(\theta)$, and after some trials, we adopt

$$\psi(\theta) = \sin 2\theta$$

and $\phi(r)$ as to verify

$$(r\phi,_r),_r - 4\phi/r = 0$$

$$(r\phi,_r),_r + 8\phi/r = -6\phi,_r = 0$$

so $\phi = Cr^2$.

$C$ is calculated using the boundary condition on $\partial S$:

$$C = (G_{22} - G_{33})/2(G_{22} + G_{33})$$

Finally,

$$\varphi = [(G_{22} - G_{33})/2(G_{22} + G_{33})]r^2 \sin 2\theta$$

and after some easy calculations,

$$\langle GJ \rangle = SR^2 \frac{1}{(1/G_{22}) + (1/G_{33})} = I_0\ 2[G_{22}\ G_{33}/(G_{22} + G_{33})]$$

## Conclusion

We found a free warping function of a beam of circular section acted upon by a torque in the direction of one of the axes of orthotropy. It occurs two shearing modulus. The result is just as it is an interesting one but may also be used for the experimental determination of shearing modulus making three tests of torsion in the three directions of orthotropy.

So, calling $X$, $Y$, $Z$ the three unknown modulus and $A$, $B$, $C$ $(\theta/2\alpha I_0)$ the three experimental results, it would be necessary to solve the nonlinear system

$$XY/(X + Y) = A, \qquad YZ/(Y + Z) = B$$

$$ZX/(Z + X) = C$$

or with the solution if

$$X = 2ABC/[B\,(A + C) - AC]$$

and $Y$ and $Z$ by cyclic permutation. Then, $X$, $Y$, $Z$ could be obtained.

The result may also be used for testing results of the finite element program on warping functions.

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# Optimal Shape Design of Lattice Structures for Accuracy

Hiroshi Furuya*
*Nagoya University, Nagoya 464-01, Japan*

## Introduction

THE purpose of this study is to present the optimal configuration of lattice space structures taking into account statistical sensitivity to shape distortions of the structures due to random member length errors. To design lattice space antennas, the length of the radio wave used for communication requires an extremely high surface accuracy. For accurate lattice structures, a new design concept is necessary to reduce structural errors. Sensitivity analyses on structural distortions provide significant information for designing tolerance errors of the structural elements and predicting the feasibility of fabricating accurate structures.

One must inevitably take into account the effects of random member length errors on the structural distortions for space structures consisting of a large number of members. Some studies based on approximate continuum analysis of the structures[1] and multiple deterministic structural analysis of the truss (the Monte Carlo method) using the finite element analysis[2] have been attempted to estimate the structural errors. Furthermore, for actively controllable structures on their deformation, the optimal locations of actuators have been studies to correct the shape distortions.[3]

## Statistical Sensitivity Analysis

In the present analysis it is assumed that structural errors can be regarded as stochastic ones and that the errors can be represented by member length tolerances. Formulations for statistical analysis on the accuracy from Ref. 4 are used herein. The formulas have been derived for expected value and variance of the structural error based on the assumption of zero-mean errors.

### Structural Deviations

A mathematical model of a three-dimensional lattice structure is defined as an assembly of straight members jointed at various nodal points. Thus, by assuming small deformation, the relationship between the member deformation vector and the corresponding nodal displacement vector can be written by the following in the matrix form:

$$dl = Adx \tag{1}$$

The equations of equilibrium can be derived from the variational principle by assuming small deformation. Thus, the equations of equilibrium at the node can be derived as follows:

$$A^TKAdx = A^TKds \tag{2}$$

where

$$K = \text{diag}(E_iA_i/l_i), \qquad (i = 1, 2, \ldots, m) \tag{3}$$

When the lattice structure is in free boundary conditions, the rank of the matrix $(A^TKA)$ is not full because the lattice structure with free boundary has rigid mode nodal displacement. Therefore, by suppressing the rigid mode deformation,

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*Assistant Professor, Structures Engineering, Department of Aeronautical Engineering.

the nodal displacement vector can be expressed in the general form as follows:

$$dx = \tilde{A}ds \tag{4}$$

**Statistical Structural Errors**

The structural error of shape distortions can be estimated by mean square error of the nodal displacement. When the expected value of the errors is assumed to be zero, the covariance matrix of the member length error vector is derived as

$$E[dsds^T] = S^2 \tag{5}$$

When the errors are uncorrelated to one another, $S$ becomes a diagonal matrix consisting of standard variations of truss members.

To consider the effects of member length errors on the reference vector $dx$, the mean square error can be defined as shown in Eq. (6):

$$\epsilon^2 = dx^TWdx = ds^{*T}S\tilde{A}^TW\tilde{A}Sds^* \tag{6}$$

Where $W$ is a positive definite and symmetric weighting matrix, and $ds^*$ is the nondimensional form of the member length error vector defined as

$$dx = \tilde{A}Sds^* \tag{7}$$

Therefore, the covariance matrix of the vector $ds^*$ becomes a unit matrix $I_m$, and $W$ can be decomposed by the Cholesky matrix $L$ as follows:

$$W = L^TL \tag{8}$$

Thus, the structural error can be directly calculated as follows:

$$\epsilon^2 = \mathrm{tr}(S\tilde{A}^TL^TL\tilde{A}S) = \mathrm{tr}(L\tilde{A}S^2\tilde{A}^TL^T) \tag{9}$$

**Sensitivity of Structural Errors**

The effects of the design parameters $\nu_\alpha$ can be calculated[5] by using Eq. (9) as

$$\epsilon^2_{,\alpha} = \sum_{k=1}^{q} \frac{\partial\lambda_k}{\partial\nu_\alpha} = \sum_{k=1}^{q} u_k^T \frac{\partial(L\tilde{A}S^2\tilde{A}^TL^T)}{\partial\nu_\alpha} u_k \tag{10}$$

where $\lambda_k$ is an eigenvalue, and $u_k$ is an eigenvector of $(L\tilde{A}S^2\tilde{A}^TL^T)$. When the design parameters are assumed to be variances of the member length errors, the following equation is derived as

$$\frac{\partial(L\tilde{A}S^2\tilde{A}^TL^T)}{\partial(\sigma_\alpha^2)} = (a'_{i\alpha}a'_{j\alpha}), \qquad (i, j = 1, 2, \ldots, q) \tag{11}$$

where $a'_{i\alpha}$ is an element of the matrix $(L\tilde{A})$. Another expression of the derivation can be yielded as follows:

$$\epsilon^2_{,\alpha} = \mathrm{tr}\left(L\tilde{A}\,\frac{\partial(S^2)}{\partial\nu_\alpha}\,\tilde{A}^TL^T\right) \tag{12}$$

Thus, once the coefficient matrix $\tilde{A}$ can be obtained, the sensitivities to structural error due to the random member length errors on the structural distortions can be calculated.

## Optimal Design

To design the optimal configuration of the lattice structures, the Levenberg-Marquardt-Morrison (LMM) method[6] is used. The procedure is summarized as follows.

In this analysis the sum of the sensitivity of all of the members is considered to be minimal as an objective function:

$$F(x) = \sum_{\alpha=1}^{m} \epsilon^2_{,\alpha} \tag{13}$$

and the design variables $x$ are the locations of the joints in the structure.

When the following vector is introduced, the problem coincides in minimizing the length of it:

$$f(x) = [\epsilon_{,1}(x), \epsilon_{,2}(x), \ldots, \epsilon_{,m}(x)]^T \tag{14}$$

To minimize the objective function, the following iteration is considered as

$$x_{k+1} = x_k + \Delta x_k \tag{15}$$

The Taylor series of $f$ can be represented by eliminating higher terms as

$$f(x_{k+1}) = f(x_k) + J(x_k)\Delta x_k \tag{16}$$

where $J$ is a Jacobian defined by the following equation:

$$J(x_k) = J_k = \begin{bmatrix} \frac{\partial\epsilon_{,1}}{\partial x_1} & \cdots & \frac{\partial\epsilon_{,1}}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial\epsilon_{,m}}{\partial x_1} & \cdots & \frac{\partial\epsilon_{,m}}{\partial x_n} \end{bmatrix}_{x=x_k} \tag{17}$$

The iteration formula for minimizing the objective function can be presented as follows:

$$(J_k^TJ_k + \nu_k^2I)\Delta x_k = -J_k^Tf(x_k) \tag{18}$$

where $\nu_k$ is called a Marquardt constant that changes dynamically in the numerical procedure. When $\nu_k = 0$, the iteration indicates the Newton-Gauss method, and when $\nu_k = \infty$, it implies the maximum gradient method.

The solution of Eq. (18) is equal to the least square solution of the following equation:

$$\begin{bmatrix} J_k \\ \cdots \\ \nu_kI \end{bmatrix} \Delta x_k = -\begin{bmatrix} f(x_k) \\ \cdots \\ 0 \end{bmatrix} \tag{19}$$

Since this equation is more desirable to be solved because of its numerical stability, it is treated as a basic equation in the procedure.

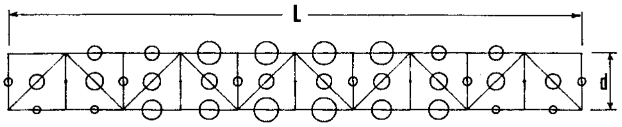
## Space Truss Structures

To establish the optimal configuration, numerical examples have been demonstrated. Two types of truss masts have been considered in the analyses. One of the masts is statically determinate, and the other is statically indeterminate. The initial configuration of the masts consists of rectangle truss modules, and the number of subdivisions is $N_s$.

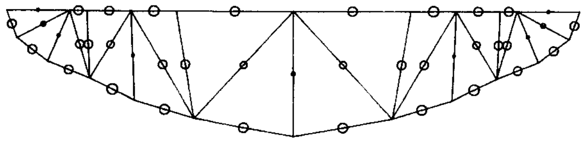
In the analyses the variation of a nondimensionalized member length error with the corresponding member length is assumed to be equal, $\sigma_\epsilon^2$, and the error is uncorrelated to each other. Design parameters are assumed to be the nondimensional member length variances. The axial stiffness and the material density of all of the members are assumed to have equal values.

In the optimization procedure, the calculation has been performed under the constraint that the upper elements form a straight line. This constraint indicates that the truss represents a two-dimensional tetrahedral antenna or a box-truss antenna. The mean square errors of the nodal displacements of the upper elements directed upward are calculated. The weighting matrix $W$ is assumed to be equal to $1/(N_s + 1)I_m$ in the analyses.

Figures 1 and 2 show the sensitivities of structural members on the structural errors for the initial and the optimal configu-

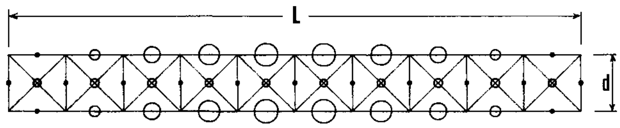



a) **Initial configuration ($d/L = 0.1$)**

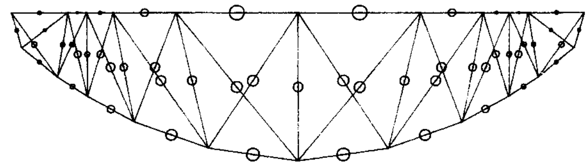


b) **Optimal configuration**

**Fig. 1 Sensitivities of two-dimensional truss for mean square error (statically determinate truss, $N_s = 10$).**




a) **Initial configuration ($d/L = 0.1$)**



b) **Optimal configuration**

**Fig. 2 Sensitivities of two-dimensional truss for mean square error (statically indeterminate truss, $N_s = 10$).**

**Table 1 Surface error ratio of optimal configuration**

| Type of truss mast | Surface error ratio ($\epsilon_{opt}^2/\epsilon_0^2$) (Nondimensional optimal error $\epsilon_{opt}^2/\sigma_\epsilon^2 L^2$) | | | |
|---|---|---|---|---|
| | Number of subdivisions, $N_s$ | | | |
| | 4 | 6 | 8 | 10 |
| Statically determinate truss | 0.233 (0.0900) | 0.330 (0.0639) | 0.478 (0.0610) | 0.548 (0.0532) |
| Statically indeterminate truss | 0.206 (0.0527) | 0.297 (0.0428) | 0.376 (0.0375) | 0.456 (0.0349) |

rations. In these figures, the area of the circle displayed on each element indicates the sensitivity levels. By comparing the statically determinate truss of the initial configuration with the statically indeterminate one, the diagonal members of the determinate truss are more sensitive for the mean square errors than those of the indeterminate one. The figures also indicate that the locations of the effective members are symmetric, and the central longitudinal elements are most effective. Thus, as for the effects of the diagonal members on the mean square errors, the statically determinate truss is more sensitive. This is why the mean square errors of the statically determinate truss are greater than those of the statically indeterminate one, although there are fewer members. These figures show that the indeterminate truss can decrease the effects of diagonal members on the mean square errors, but for the contribution of the longitudinal member on the errors, the difference between the indeterminate truss and the determinate one is very small for the initial configurations. For the optimal configuration, it is shown that the effects of the longitudinal member on the square errors can be reduced, and therefore, the optimal shape is less sensitive in changing the element lengths.

Table 1 shows ratios of the nondimensional mean square errors of upper surface displacement of the initial and the optimal truss masts vs the number of subdivisions of the initial truss mast. For the initial configuration, the errors ($\epsilon_0$) have been calculated under the same truss depth ($d/L = 0.1$). These results indicate that the optimal configuration is more effective for reducing the structural error in the case of fewer subdivisions, although the mean square error of the optimal configuration increases as the number of subdivisions decreases. For the efficiency of optimization, the statically indeterminate truss is better than the statically determinate one.

## Concluding Remarks

The optimal shape design for structural accuracy of space structures based on a sensitivity analysis has been presented. The results of the two-dimensional truss masts have been demonstrated to predict the effective members contributing to the errors, and the characteristics of the optimal configuration have been investigated.

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# Displacement Approximations for Optimization of Beams Defined in Nonprincipal Coordinate Systems

P. A. Fenyes* and R. V. Lust†
*General Motors Research Laboratories, Warren, Michigan 48090*

## Introduction

APPROXIMATION concepts have been instrumental to many of the advances in structural optimization.[1,2] Early structural optimization methods used finite element analysis to evaluate the structural response throughout the optimization procedure. This approach required large numbers of analyses and was costly, particularly for large problems. By approximating the structural response, it became possible to eliminate many of these costly finite element analyses. The earliest approximations were in the form of first-order Taylor series, expanding the responses directly in terms of the design variables or their reciprocals. In truss structures, where the

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*Senior Research Engineer, Engineering Mechanics Department, 256 RMB, 30500 Mound Road.

†Staff Research Engineer, Engineering Mechanics Department, 256 RMB, 30500 Mound Road. Senior Member AIAA.